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Quantal Information Theory

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Information to be gained from the measurement of observables in many areas of physics can be represented by the concatenation of Hermitian, pseudo-Hermitian, and real qubits to form a quantal analogs of the 'tape' of a Turing machine. There is a distinction between selected information, which is in principle predictable, and unselected information, which is indeterminate. Qubits and tapes are subject to transformations which may be used to convert them to selected form. It is shown how to represent information derived from quantum mechanics, quantum statistics, particle physics, and cosmology in this way.

1. INTRODUCTION

The storage, processing, and output of information, according to Turing (1936), requires a 'tape' consisting of a sequence of two-valued bits and a 'machine' that scans and modifies the tape in a deterministic manner. When the initial state of the tape is given, the output of such a machine, typical of most contemporary computers, is completely predictable in principle, and, with Shannon and Weaver's (1949) definition, the information to be gained from it is zero. However, a significant generalization of the Turing machine, operating on a quantal analog of Turing's tape, was described by Benioff (1980, 1982), and in more recent years there has been considerable interest in the possibility of quantum computation (Hirota *et al.*, 1997), especially since the work of Deutsch and Josza (1992) and Schor (1994) suggesting that it could facilitate the performance of some tasks very much more efficiently than is possible at present. But the outcome of quantal information processing is not predictable in general, and there are difficulties in the implementation of quantum computing (see, *e.g.*, Barenco, 1996), of which

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those associated with preparing and scanning a quantal tape have come nearest to resolution (Zak and Willliams, 1998).

It is remarkable that for a long time following the important work of Brillouin (1964) the implications of quantal information theory received little attention, in spite of their potential importance for the understanding of theoretical physics and quantum mechanics in particular. Often unnoticed, though essential for areas such as quantal computing, is the distinction between selected information, which is in principle predictable, as in the detection of a particle (de Broglie, 1959), and unselected information, which is not uncertain in the sense of classical information theory, but indeterminate. Also, in the literature there are physical hypotheses which are supported by a considerable amount of experimentally derived information, and others concerning which no information is even possible.

This paper will be concerned principally with the representation in a standard form, comparable with Turing's tape, of the quantal information associated with physical observables in the context of theoretical physics. It will have some relevance to quantal computing, since in all instances similarity transformations will be obtained which convert the quantal information to selected form. But information theory illuminates a variety of problems of contemporary interest, some of which, such as string theory, the theory of consciousness, and the geometry of space-time, can only be referred to in passing within the scope of the paper.

The intention of the following is to give a useful definition of quantal information, and to discuss in a general context the representation of quantal information in terms of elementary qubits and the creation and assembly of qubits to form the different kinds of observables at present recognized by physics. There are three fundamental types of qubit, which may be characterized as Hermitian, pseudo-Hermitian, and real, with two different pseudo-Hermitian varieties. Each type has an irreducible two-dimensional matrix representation, within which corresponding creation and annihilation operators may be defined. It will be shown that information derived from almost every branch of modern physics may be represented as quantal 'tapes' consisting of qubits of different types, but with rather simple structures and symmetries.

2. REPRESENTATION OF QUANTAL INFORMATION

In quantum mechanics, any observable may be represented by a matrix A with a set of real eigenvalues A_r which are possible values resulting from a measurement of the observable. In spectral form,

$$A = \sum_r A_r g_r \tag{1}$$

where the g_r constitute a complete set of minimal projections. The eigenvalues may be ordered so that $A_r \leq A_s$ when $r < s$, and the summation Σ_r may be interpreted as an integration $\int dr$ if the eigenvalues are continuous. The subscript r can always be expressed as a binary number:

$$r = \sum_j 2^j r_j \qquad (r_j^2 = r_j) \tag{2}$$

where j takes nonnegative values if the spectrum is discrete, but negative values if the spectrum is at least partly continuous. Selected observables, whose measured values are in principle predictable, may be represented by diagonal matrices. In the stationary states of an isolated system, and also in the states of a system in equilibrium with its environment, the energy is a selected observable.

The projections g_r are observables with a single nonvanishing eigenvalue 1, and are required to be Hermitian if the measurement is in the inertial frame of the observer. The expectation value $\langle A \rangle$ of A is $\Sigma_r A_r p_r$, where p_r is the (nonvanishing) probability that the measurement of A will yield the value A_r. In the following, the information I to be gained from the measurement of A will be treated as an observable, with eigenvalues $I_r = -\log p_r$:

$$I = -\sum_r \log(p_r) g_r \tag{3}$$

The maximum amount of information to be gained from the measurement of a selected observable of a system is connected by the relation $I = -\log P$ with the statistical matrix P of von Neumann (1932). But since $p_r = \langle g_r \rangle$ even if the observable is not selected, I has the expectation value

$$\langle I \rangle = -\sum_r \log(p_r) p_r \tag{4}$$

which is Shannon's classical definition of information to be gained. A conscious observer may become aware of the result of the measurement of an observable by quantal processes described in detail elsewhere (Green and Triffet, 1997a, b).

We shall show in the following how the information content of any observable may be reduced to quantal bits. A qubit may be defined, by analogy with its classical counterpart, as an observable with two eigenvalues 0 and 1. As such it is an elementary projection, represented by a matrix n of the second degree; it satisfies the characteristic identity $n^2 = n$ and has unit trace and determinant zero:

$$\text{tr}(n) = n_{11} + n_{22} = 1, \qquad \det(n) = n_{11}n_{22} - n_{12}n_{21} = 0 \tag{5}$$

If $p = \langle n \rangle$ is the expectation value of n, the information I_n gained by the measurement of n is given by

$$I_n = -\log(p)n - \log(p^c)n^c, \qquad n^c = 1 - n, \quad p^c = 1 - p \tag{6}$$

where n^c is the complement of n, etc.

A matrix such as n can be presented as a tensor product of spinor and cospinor factors:

$$n = \phi\bar{\phi}, \qquad \bar{\phi}\phi = 1 \tag{7}$$

where $\bar{\phi}\phi$ denotes the scalar product. The factorization can be made unique by specifying the components

$$\phi_1 = n_{12}/(n_{22})^{1/2}, \qquad \phi_2 = \bar{\phi}_2 = (n_{22})^{1/2}, \qquad \bar{\phi}_1 = n_{21}/(n_{22})^{1/2} \tag{8}$$

Similarly, for the complementary qubit n^c,

$$n^c = \phi^c\bar{\phi}^c, \qquad \bar{\phi}^c\phi^c = 1, \qquad \bar{\phi}^c\phi = \bar{\phi}\phi^c = 0 \tag{9}$$

The matrix n representing a qubit can also be expressed in terms of a vector matrix ξ, formed from three anticommuting matrices τ_a of order 2, and a real unit vector ξ^a ($a = 1, 2, 3$):

$$n(\xi) = \tfrac{1}{2}(1 + \xi), \qquad \{\tau_a, \tau_b\} = 2\eta_{ab}, \qquad \eta_{ab}\xi^a\xi^b = 1 \tag{10}$$

where η_{ab} is the informational metric tensor in three dimensions. Two qubits $n = n(\xi)$ and $n' = n(\xi')$ are of the same type and variety if they are connected by a similarity transformation $n \to n' = unu^{-1}$ generated by the commutator $[\xi', \xi]$:

$$n(\xi') = u(\xi', \xi)n(\xi)u^{-1}(\xi', \xi), \qquad u(\xi', \xi) = \exp(\tfrac{1}{4}[\xi', \xi]) \tag{11}$$

Assuming that τ_3 is diagonal, if ξ' has components $\xi'^a = \delta_3^a$, then the components of ξ^a are identified as the parameters of the particular transformation $u(\delta_3, \xi)$ required to convert $n(\xi)$ to selected form, in which the result of a measurement of the qubit is in principle determinate. But in general $u(\xi', \xi)$ belongs to a continuous group of a type depending on the informational metric, and there are three different types:

(1) If the metric is Euclidean, then (τ_1, τ_2, τ_3) are Pauli matrices; the group of transformations is $SU(2)$ and we shall refer to the qubits as Hermitian.

(2) If the metric is pseudo-Euclidean with signature (-1, 1, -1), then ($i\tau_1$, τ_2, $i\tau_3$) are Pauli matrices; the group of transformations is $SU(1, 1)$ and we shall refer to the qubits as pseudo-Hermitian. There are two different varieties of pseudo-Hermitian qubits, distinguished by the sign of ξ^2; $n\tau_2/\xi^2$ is a Hermitian qubit.

(3) If the metric is pseudo-Euclidean with signature $(1, -1, 1)$, then $(\tau_1, i\tau_2, \tau_3)$ are Pauli matrices; the group of transformations is $Sl(2, r)$ and we shall refer to the qubits as real.

We shall give some examples of each type.

The number of fermions of a particular type may be represented by a Hermitian qubit. The corresponding fermion creation and annihilation matrices $\bar{c}$ and c satisfying

$$\bar{c}c = n, \qquad c\bar{c} = n^c, \qquad \bar{c}^2 = c^2 = 0 \tag{12}$$

can be defined as outer (matrix) products of the spinor and cospinor factors of n and n^c:

$$\bar{c} = \phi\overline{\phi}^c, \qquad c = \phi^c\overline{\phi} \tag{13}$$

A projection for the spin of a system of spin half is a Hermitian qubit, and the vector ξ^a determines the direction of the spin. The transformation (11) corresponds to a rotation of the inertial frame of the observer, and the spin angular momentum of the system (in units of $\hbar$) is $-\frac{1}{4}i[\tau_a, \tau_b]$. Similar considerations apply naturally for isospin.

If

$$\xi^1 = \pm k^1/(mc), \qquad \xi^2 = \pm E/(mc^2), \qquad \xi^3 = \pm k^3/(mc) \tag{14}$$

where E is the energy of a system of mass m and k^a is its momentum in the plane of its motion (so that $k^2 = 0$), then the qubit is pseudo-Hermitian and represents the state of motion of a particle in the momentum representation. The sign determines the variety; the positive sign is normally chosen for particles and the negative sign for antiparticles. In this instance the transformation (11) corresponds to a Lorentz transformation of the inertial frame of the observer.

A point of a two-dimensional subspace of de Sitter space with indefinite metric and space-time coordinates $x^a = R\xi^a$ can be represented by a real qubit. Then the transformation (11) in general includes a translation in space-time as well as a Lorentz transformation of the inertial frame of the observer.

A point on the surface of a string in string theory (see, *e.g.*, Lüst and Theisen, 1989) can also be represented by a real qubit. Locally, a string is usually pictured as cylindrical, but a hyperboloid of one sheet has the same topology as a cylinder and all points are equivalent in de Sitter space. The action associated with a string between the times 0 and x^2 is identified with its surface area and has the value

$$A = -m \int_0^{x^2} [1 - (dx^1/dx^2)^2]^{1/2}\, dx^2 \tag{15}$$

corresponding to the Lagrangian $m[1 - (dx^1/dx^2)^2]^{1/2}$, as required by the special theory of relativity. There is a reciprocal relation between pseudo-

Hermitian and real qubits, illustrated by the identification in an unquantized theory of the energy-momentum k^a in (14) with $m\, dx^a/ds$, where $ds^2 = \eta_{ab}\, dx^a\, dx^b$.

3. QUANTAL TAPES

A natural generalization of the qubit is the quantal tape obtained by the concatenation of a finite number or a countable infinity of qubits. Such a tape may be represented as the direct product of the matrices $n^{[1]}$, $n^{[2]}$, $n^{[3]}$, . . . , representing the separate qubits:

$$n = n^{[1]} \otimes n^{[2]} \otimes n^{[3]} \ldots = n^{(1)}\, n^{(2)}\, n^{(3)} \ldots \tag{16}$$

where the commuting factors $n^{(r)}$ of n, representing the elements of the tape, are

$$n^{(1)} = n^{[1]} \otimes 1 \otimes 1 \ldots, \qquad n^{(2)} = 1 \otimes n^{[2]} \otimes 1 \ldots, \qquad n^{(3)} = 1 \otimes 1 \otimes n^{[3]} \ldots. \tag{17}$$

The tape is subject to transformations of the type shown in (11), and there is a particular transformation of that type which converts the tape to selected form. As in (12), each element of the tape can be factorized,

$$n^{(i)} = \overline{c}^{(i)}\, c^{(i)} = 1 - c^{(i)}\, \overline{c}^{(i)}, \qquad \overline{c}^{(i)2} = c^{(i)2} = 0 \tag{18}$$

but, unlike different fermion creation and annihilation operators, factors with different superscripts j commute with one another.

The direct product n is a minimal projection, belonging to a complete commuting orthogonal set defined by

$$g_r = \prod_j\, (c^{(j)})^{r_j}\, n\, (\overline{c}^{(j)})^{r_j} \tag{19}$$

A set of minimal projections g_r of this type can be used to construct any observable A, as shown in (1). According to (3), the information I to be gained from the quantal tape is the sum of the information to be gained from the separate qubits:

$$I = \sum_j I^{(j)} = \sum_j\, [\log(p^{(j)})n^{(j)} + \log(p^{(j)c})n^{(j)c}] \tag{20}$$

where $p^{(j)} = \langle n^{(j)} \rangle = 1 - p^{(j)c}$.

4. QUANTUM STATISTICS

If unit vectors $\xi^{(j)}$ are defined as in (10),

$$n^{(j)} = \tfrac{1}{2}(1 + \xi^{(j)}) \tag{21}$$

and the factors $\bar{c}^{(j)}$ and $c^{(j)}$ by (18), a set of fermion creation and annihilation matrices $\bar{f}^{(j)}$ and $f^{(j)}$ may be defined by

$$\bar{f}^{(j)} = \left(\prod_{k<j} \xi^{(k)}\right) \bar{c}^{(j)}, \qquad f^{(j)} = c^{(j)} \left(\prod_{k<j} \xi^{(k)}\right) \tag{22}$$

A quantal tape of indefinite length can also be used to represent a boson number in the form

$$N = \sum_j 2^j n^{(j)} \qquad (j \geq 0) \tag{23}$$

where each $n^{(j)}$ is a qubit with eigenvalues $n_0^{(j)} = 0$ and $n_1^{(j)} = 1$. In a measurement of N, the eigenvalue $n_k^{(j)}$ $(k = 0$ or $1)$ of $n^{(j)}$ is the $(j + 1)$th binary digit of the measured value $N_k = \Sigma_j\, 2^j n_k^{(j)}$. The corresponding creation and annihilation matrices $\bar{b}$ and b, satisfying $b\bar{b} - \bar{b}b = 1$ and $\bar{b}b = N$, can be expressed in terms of the factors $\bar{c}^{(j)}$ and $c^{(j)}$ of (18),

$$\bar{b} = \sum_j \lambda_j^{1/2} \left(\prod_{k<j} c^{(k)}\right) \bar{c}^{(j)}, \qquad b = \sum_j \lambda_j^{1/2} c^{(j)} \left(\prod_{k<j} \bar{c}^{(k)}\right) \tag{24}$$

with coefficients $\lambda_k^{1/2}$ depending on the $n^{(j)}$ with $j > k$:

$$\lambda_k = 2^k + \sum_{j>0} 2^{k+j}\, n^{(k+j)} \tag{25}$$

We thus obtain representations of the bosonic matrices in terms of fermionic creation and annihilation matrices $\bar{c}^{(j)}$ and $c^{(j)}$ affecting individual constituents of the quantal tape. By an obvious extension we obtain the corresponding representations

$$N_\alpha = \sum_j 2^j n_\alpha^{(j)} = \bar{b}_\alpha b_\alpha$$

$$\bar{b}_\alpha = \sum_j \lambda_{j,\alpha}^{1/2} \left(\prod_{k<j} c_\alpha^{(k)}\right) \bar{c}_\alpha^{(j\cdot r)}, \qquad b_\alpha = \sum_j \lambda_{j,\alpha}^{1/2}\, c_\alpha^{(j)} \left(\prod_{k<j} \bar{c}_\alpha^{(k)}\right)$$

$$\lambda_{k,\alpha} = 2^k + \sum_{j>0} 2^{k+j}\, n_\alpha^{(k+j)} \tag{26}$$

for any number of bosons, in terms of the factors $\bar{c}_\alpha^{(j)}$ and $c_\alpha^{(j)}$ of a countable set of qubits $n_\alpha^{(j)}$.

A set of parafermion creation and annihilation matrices of order p can be defined in terms of the mp matrices $\overline{c}_j^{(u)}$ and $c_j^{(u)}$ ($j = 1, 2, \ldots, m$; $u = 1, \ldots, p$) obtained by the factorization $n_j^{(u)} = \overline{c}_j^{(u)} c_j^{(u)}$ of the commuting projections $n_j^{(u)}$ constituting a quantal tape. The matrices

$$\overline{e}_j^{(u)} = \overline{c}_j^{(u)} \left(\prod_{k<j} \xi_k^{(u)} \right), \qquad e_j^{(u)} = c_j^{(u)} \left(\prod_{k<j} \xi_k^{(u)} \right) \qquad (\xi_k^{(u)} = 2n_k^{(u)} - 1) \tag{27}$$

commute for different values of u, but satisfy anticommutation relations

$$\begin{aligned} \{e_j^{(u)}, \overline{e}_k^{(u)}\} &\equiv e_j^{(u)}\overline{e}_k^{(u)} + \overline{e}_k^{(u)}e_j^{(u)} = \delta_{jk}, \\ \{e_j^{(u)}, e_k^{(u)}\} &\equiv e_j^{(u)}e_k^{(u)} + e_k^{(u)}e_j^{(u)} = 0 \\ \{\overline{e}_j^{(u)}, \overline{e}_k^{(u)}\} &\equiv \overline{e}_j^{(u)}\overline{e}_k^{(u)} + \overline{e}_k^{(u)}\overline{e}_j^{(u)} = 0 \end{aligned} \tag{28}$$

so that, as in the author's original paper (Green, 1953), the parafermion creation and annihilation matrices can be defined by

$$e^j = \sum_{u=1}^{p} \overline{e}_j^{(u)}, \qquad e_j = \sum_{u=1}^{p} e_j^{(u)} \tag{29}$$

The number m_j of parafermions of the jth type is the observable given by

$$m_j = \tfrac{1}{2}([e_j, e^j] + p) = \tfrac{1}{2}\sum_{u=1}^{p} ([\overline{e}_j^{(u)}, e_j^{(u)}] + 1) \tag{30}$$

and has integral eigenvalues extending from 0 to p, as required.

As noted by Ryan and Sudarshan (1963), matrix representations of $so(m + 1)$ can be obtained with elements expressed in terms of the conjugate elements e^j and e_j and

$$e^j_k = \tfrac{1}{2}[e^j, e_k] = \tfrac{1}{2}\sum_{u=1}^{p} [\overline{e}_j^{(u)}, e_k^{(u)}], \qquad e^{jk} = \tfrac{1}{2}[e^j, e^k] = \tfrac{1}{2}\sum_{u=1}^{p} [\overline{e}_j^{(u)}, \overline{e}_k^{(u)}]$$

$$e_{jk} = \tfrac{1}{2}[e_j, e_k] + \tfrac{1}{2}p\delta_{jk} = \tfrac{1}{2}\sum_{u=1}^{p} ([e_j^{(u)}, e_k^{(u)}] + \tfrac{1}{2}\delta_{jk}) \tag{31}$$

The Lie algebra so defined has subalgebras $so(2m)$ with elements e^j_k, e^{jk}, and e_{jk} and $su\,(m)$ with elements e^j_k, while $u\,(2m)$ has the subalgebra $sp(2m)$ with elements $e^j_k + e^{j+N}_{k+N}$, $e^j_{k+N} + e^k_{j+N}$, and $e^{j+N}_k + e^{k+N}_j$, so that representations of all the classical Lie algebras can be obtained in terms of qubits in this way. For the exceptional algebras, 27 pairs of parafermion creation and annihilation matrices $e^{(j,k,l)}$ and $e_{(j,k,l)}$ ($1 \le j, k, l \le 3$) are required for E_6, 45 pairs $e^{(jk,l)}$ and $e_{(jk,l)}$ ($1 \le j \ne k \le 6$, $1 \le l \le 3$) for E_7, 84 pairs $e^{(jkl)}$

and $e_{(jkl)}$ $(1 \leq j \neq k \neq l \leq 9)$ for E_8, 18 pairs $e^{(jk,l)}$ and $e_{(jk,l)}$ $(1 \leq j, k \leq 3, 1 \leq l \leq 3)$ required for F_4, and 3 pairs e^j and e_j $(1 \leq j \leq 3)$ required for G_2. With this notation, the elements of the exceptional algebras are as follows:

E_6: $e^{(j,b,c)}_{(k,b,c)}$, $e^{(a,j,c)}_{(a,k,c)}$, $e^{(a,b,j)}_{(a,b,k)}$, $e^{(j,k,l)} + \varepsilon^{jax}\varepsilon^{kby}\varepsilon^{lcz}e_{(a,b,c)(x,y,z)}$ and conjugates

E_7: $e^{(jb,x)}_{(kb,x)}$, $e^{(ab,j)}_{(ab,k)}$, $e^{(jk,l)} + \frac{1}{3}\varepsilon^{jkabcd}\varepsilon^{lyz}e_{(ab,y)(cd,z)}$ and conjugates

E_8: $e^{(jbc)}_{(kbc)}$, $e^{(jkl)} + (1/40)\varepsilon^{jklabcxyz}e_{(abc)(xyz)}$ and conjugates

F_4: $e^{(jb;x)}_{(kb;x)}$, $e^{(ab;j)}_{(ab;k)}$, $e^{(jk;l)} + \frac{1}{3}\varepsilon^{jab}\varepsilon^{kcd}\varepsilon^{lyz}e_{(ac;y)(bd;z)}$ and conjugates

G_2: e^j_k, $e^j + \varepsilon^{jkl}e_{kl}$ and conjugates

Representations of superalgebras such as $osp(m, m')$ can be constructed in a similar way from parfermion and paraboson creation and annihilation matrices.

5. NONRELATIVISTIC QUANTUM MECHANICS

The construction already given for boson creation and annihilation matrices $\bar{b}_\alpha$ and b_α allows the representation of any number of canonical coordinate and momentum observables q_α and p_α satisfying the canonical commutation relation $q_\alpha p_\beta - p_\beta q_\alpha = i\delta_{\alpha\beta}$, thus

$$q_\alpha = 2^{-1/2}(b_\alpha + \bar{b}_\alpha), \qquad p_\alpha = 2^{-1/2}i(\bar{b}_\alpha - b_\alpha) \tag{32}$$

These observables have a continuum of eigenvalues with eigenvectors in the countably infinite-dimensional representation of the boson matrices. More typically, the eigenvalues of some function $A = A(q, p)$ of the canonical observables, such as the Hamiltonian energy, are required, and these can be obtained by a factorization method which makes use of the information-theoretic representation of the observable.

The method relies on the sequential construction of the commuting observables

$$A^{(r)} = A_r g_1 + A_{r+1} g_2 + A_{r+2} g_3 + \ldots \qquad (r = 1, 2, 3, \ldots) \tag{33}$$

starting with $A^{(1)} = A$. The g_r are projections corresponding to the eigenvalues A_r, and are supposed to be expressed as shown in (19), from which it follows that any pair may be factorized in the form

$$g_r = h_{rs}h_{sr}, \qquad g_s = h_{sr}h_{rs}, \qquad h_{rs} = \prod_j (c^{(j)})^{r_j n} \prod_k (\bar{c}^{(k)})^{r_k} \tag{34}$$

When the g_r are diagonal, the h_{r+1r} are codiagonal matrices.

Since $A^{(r)} - A_r$ is positive definite but has a vanishing lowest eigenvalue, this matrix can be factorized as $A^{(r)} - A_r = \bar{c}_r c_r$, where

$$\bar{c}_r = (A_{r+1} - A_r)^{1/2} h_{r+1,r} + (A_{r+2} - A_r)^{1/2} h_{r+2,r+1} + (A_{r+3} - A_r)^{1/2} h_{r+3,r+2} + \cdots \tag{35}$$

and is the conjugate of c_r. Then the eigenvalues A_r together with the factors $\bar{c}_r$, c_r, and the $A^{(r+1)}$ are successively determined by the relations

$$A^{(r)} = \bar{c}_r c_r + A_r, \qquad A^{(r+1)} = c_r \bar{c}_r + A_r \tag{36}$$

The factors $\bar{c}_r$ and c_r of $A^{(r)} - A_r$ in (35) are not unique, and may be replaced by other matrices $\bar{c}_r\bar{u}$ and uc_r if $\bar{u}$ and u satisfy the unitary or pseudounitary condition $\bar{u}u = 1$, but this change does not affect the eigenvalue. The particular factors chosen above ensure that $A^{(r+1)}$ commutes with $A^{(r)}$, but that feature is not essential for the efficacy of the factorization method and any sequence of factorizations consistent with (36) will yield the same eigenvalues. However, it should be noted that there are normally at least two values of A_r which allow the matrix $A^{(r)} - A_r$ to be factorized into conjugate matrices $\bar{c}_r$ and c_r, and it is necessary to choose the *greater* of these values when the eigenvalues are in ascending order. There are obviously two possibilities: the eigenvalues may increase indefinitely as $r \to \infty$, or they may approach a limiting value; in the latter event, there is normally a continuum of eigenvalues beyond the limit.

We note also that from (44) it follows that $A^{(s+1)}c_s = c_s A^{(s)}$, so that

$$\bar{c}_0 \bar{c}_1 \ldots \bar{c}_r c_r \ldots c_1 c_0 = (A - A_0)(A - A_1) \ldots (A - A_r) \tag{37}$$

with $r + 1$ vanishing eigenvalues.

A simple example is provided by the determination of the energy levels of two bodies with an attractive Coulomb potential e_+e_-/q together with a weak constraining force increasing linearly with separation to resolve the continuum. The Hamiltonian energy of the relative motion is given by $H = A/2m$, where m is the reduced mass of the system and

$$A = p^2 - l(l+1)/q^2 + 2me_+e_-/q + \gamma^2 q^2 \tag{38}$$

where l is the orbital quantum number and $qp - pq = i$. The factors of $A - A^{(0)}$ in particular and $A - A^{(r)}$ in general are easily seen to be given by

$$\bar{c}_r = p - i(\alpha_r + \beta_r/q - \gamma_r q), \qquad c_r = p + i(\alpha_r + \beta_r/q - \gamma_r q) \tag{39}$$

with $\alpha_0^2 = -A_0$, $\beta_0 = l + 1$, and $\gamma_0 = \gamma$. The eigenvalues $H_r = A_r/2m$ of the energy are obtained from the relations (44), which yield

$$A_r = -\alpha_r^2 + (2\beta_r + 1)\gamma_r, \qquad \alpha_r\beta_r = -me_+e_-,$$
$$\beta_r = l + r + 1, \qquad \alpha_r\gamma_r = \alpha_0\gamma \tag{40}$$

For very small values of γ the lower negative values of H_r approximate closely to the energy levels of a hydrogenlike atom, but the positive values corresponding to ionized states are very closely spaced and approximate to a continuum.

6. REPRESENTATIONS OF THE DE SITTER GROUP

The simplest representations on a quantal tape of the Lorentz group, including rotations in three dimensions, are on combinations of Hermitian and pseudo-Hermitian qubits. The rotations for a system of spin $\frac{1}{2}\sigma$ can be represented on a segment $n_\sigma(\xi)$ of the tape consisting of a product of σ identical Hermitian qubits:

$$n_\sigma(\xi) = \prod_r n^{(r)}(\xi) \qquad (r = 1, \ldots, \sigma)$$

and depending on single Euclidean vector ξ^a. The transformation $n_\sigma(\xi) \rightarrow n_\sigma(\xi')$ is effected by a unitary matrix

$$v(\xi', \xi) = \prod_r \exp(-\tfrac{1}{4}[(\xi^{(r)'}, \xi^{(r)})], \qquad \xi^{(r)} = \xi^a \sigma_a^{(r)} \tag{41}$$

and the components s_{ab} of the spin angular momentum of the system, defined as generators of rotations, are therefore given by

$$S_{ab} = -\tfrac{1}{4} i \sum_j [\sigma_a^{(r)}, \sigma_b^{(r)}] \tag{42}$$

The representation on the tape is completely symmetric in the sets of Pauli matrices $\sigma_a^{(r)}$, so that states with spin less than $\frac{1}{2}\sigma$ are excluded.

Lorentz transformations are represented in a similar way on a segment $n_\tau(\tilde{\xi})$ of the tape consisting of a product of another set of σ pseudo-Hermitian qubits

$$n_\tau(\tilde{\xi}) = \prod_r n^{(\sigma+r)}(\tilde{\xi}), \qquad n^{(\sigma+r)}(\tilde{\xi}) = \tfrac{1}{2}(1 + \tilde{\xi}^{(r)}), \qquad \tilde{\xi}^{(r)} = \tilde{\xi}^a \tau_a^{(r)} \tag{43}$$

where the $(i\tau_1^{(r)}, \tau_2^{(r)}, i\tau_3^{(r)})$ are sets of Pauli matrices, so that the vector $\tilde{\xi}^a$ is pseudo-Euclidean. The Lorentz transformations are generated by matrices $\tilde{s}_{ab}$ given by

$$\tilde{s}_{ab} = -\tfrac{1}{4} i \sum_j [\tilde{\tau}_a^{(r)}, \tilde{\tau}_b^{(r)}] \tag{44}$$

As already noticed, the vector $\tilde{\xi}^a$ may be interpreted as the energy-momentum of a system, and the sign of the component $\tilde{\xi}^2$ determines whether the system is regarded as consisting of matter or antimatter. The momentum is restricted to a plane, so that the Lorentz transformations include rotations about only one axis. However, the tape $n = n_\sigma n_\tau$ obtained by combining the Hermitian and pseudo-Hermitian segments carries representations of a complete set of motions in a de Sitter space, which is four-dimensional and approximates closely to special relativistic space-time in any small region. The factor $n^{(r)}n^{(\sigma+\tau)}$ of n may also be written

$$n^{(r)}n^{(\sigma+r)} = n^{(r)}q^{(r)}, \qquad q^{(r)}(v) = \tfrac{1}{2}(1 + v^j\gamma_j^{(r)} \qquad (j = 0, 1, 2, 3, 4)$$

$$v^j = (\tilde{\xi}^2, \tilde{\xi}^1\tilde{\xi}^1, \tilde{\xi}^1\tilde{\xi}^2, \tilde{\xi}^1\tilde{\xi}^3, \tilde{\xi}^3)$$

$$\gamma_j^{(r)} = (\tau_2^{(j)}, \tau_1^{(j)}\sigma_1^{(j)}, \tau_1^{(j)}\sigma_2^{(j)}, \tau_1^{(j)}\sigma_3^{(j)}, \tau_3^{(j)}) \tag{45}$$

where the $\gamma_j^{(r)}$ are sets of Dirac matrices (in the usual notation, except that $\gamma_4^{(r)} = i\gamma_5^{(r)}$) and the v^j are projective components of a relativistic velocity related to the energy-momentum by

$$v^j = \pm k^j/m, \qquad \eta_{jk}v^jv^k = 1 \tag{46}$$

where η_{jk} has the signature $(1, -1, -1, -1, -1)$. The component k^4 of k^j is orthogonal to the space-time surface in de Sitter space, and is therefore very near to zero for any system in a local inertial frame.

If the spin and generalized Dirac matrices of the system are denoted by

$$s_a = \tfrac{1}{2}\sum_r \sigma_a^{(r)}, \qquad \alpha_j = \sum_t \gamma_j^{(r)} \tag{47}$$

the tape n representing the system satisfies the equations

$$\xi^a s_a n = sn, \qquad k^j\alpha_j n = mn \tag{48}$$

The tape is therefore a relativistic density matrix for a free system in a pure state, and can be factorized in the form

$$n = \chi\bar{\chi}, \qquad \chi = \prod_r \phi^{(r)}, \qquad \bar{\chi} = \prod_r \bar{\phi}^{(r)} \tag{49}$$

where $n_\sigma^{(r)}n_\tau^{(r)} = \phi^{(r)}\bar{\phi}^{(r)}$. The spinors χ and $\bar{\chi}$ are then of the type commonly used in relativistic quantum mechanics.

If real, instead of pseudo-Hermitian qubits are used, the resulting tape may be used to encode information concerning the position of an event in space-time, typically the point of emission or absorption of a particle. When the real qubits are substituted for the pseudo-Hermitian qubits in (45), we have

$$n^{(r)}n^{(\sigma+r)} = n^{(r)}q^{(r)}(y), \qquad q^{(r)}(y) = \tfrac{1}{2}(1 + iy^j\gamma_j^{(r)}) \qquad (j = 0, 1, 2, 3, 4) \tag{50}$$

so that $\eta_{jk}y^jy^k = 1$ with a metric that now has the signature $(-1, 1, 1, 1, 1)$. If $x^j = Ry^j$, the x^j are projective coordinates in a de Sitter space of curvature R^{-1}, which approximate closely to the space-time coordinates of special relativity in any neighborhood that is small compared with R. If we write

$n = n(y)$, the transformation $n(y) \rightarrow n(y') = u(y', y)n(y)u^{-1}(y', y)$ is effected by the matrix

$$u(y', y) = \prod_r \exp(-\tfrac{1}{4}y^j y'^k[\gamma_j^{(r)}, \gamma_k^{(r)}]) = \exp(\tfrac{1}{2}iy^j y'^k \alpha_{jk})$$

$$\alpha_{jk} = \tfrac{1}{2}i[\alpha_j, \alpha_k] \tag{51}$$

The α_{jk} are generators of the de Sitter group in a symmetric representation, satisfying

$$[\alpha_{jk}, \alpha_l] = i(\eta_{kl}\alpha_j - \eta_{jl}\alpha_k)$$

$$[\alpha_{jk}, \alpha_{lm}] = i(\eta_{kl}\alpha_{jm} - \eta_{jl}\alpha_{km} - \eta_{km}\alpha_{jl} + \eta_{jm}\alpha_{kl}) \tag{52}$$

They can all be interpreted as fundamental physical observables of a system represented by the tape. The energy E, the momentum $\mathbf{P}$, the angular momentum $\mathbf{J}$, and the central vector $\mathbf{C} = M\mathbf{Q}$ (where M is the mass and $\mathbf{Q}$ is the position vector of the centre of mass) of the system in arbitrary units may be defined by

$$E = \hbar c\alpha_{04}/R, \qquad (P_1, P_2, P_3) = \hbar(\alpha_{14}, \alpha_{24}, \alpha_{34})/R$$

$$(J_1, J_2, J_3) = \hbar(\alpha_{23}, \alpha_{31}, \alpha_{12}), \qquad (C_1, C_2, C_3) = \hbar(\alpha_{01}, \alpha_{02}, \alpha_{03})/c \tag{53}$$

Because c and R are both very large in comparison with observables with the same dimensions, components of both $\mathbf{Q}$ and $\mathbf{P}$ very nearly commute, and as $E/(Mc^2) \approx 1$, the commutation relations $Q_aP_b - P_bQ_a = i\hbar\delta_{ab}$ of nonrelativistic quantum mechanics are also satisfied to a very good approximation.

As generators of transformations of the tape representing a system occupying a region V, the fundamental observables are additive on the region. For a system in equilibrium with its environment, the information I to be gained is also an additive function, and a linear relation of the type

$$I = -\beta(E + W - \mathbf{u} \cdot \mathbf{P} - \omega \cdot \mathbf{J} - \Sigma\, \chi_a N_a) \tag{54}$$

must therefore hold, where $W = \int p\, dV$ is the work function, N_a is the number of particles of the ath type in the region, and the parameters have a well-known thermodynamic significance. This relation may be used to justify Shannon's identification of the classical information $\langle I \rangle$ with the entropy, in suitable units.

7. GRAVITATIONAL AND COSMOLOGICAL EFFECTS

We now consider the more general use of real qubits in the coding of information concerning events in a Riemannian space-time consistent with

Einstein's theory of gravitation and cosmologies more complex than those based on de Sitter space. Each event is assigned discrete values of a set of coordinates x^λ ($\lambda = 0, 1, 2, 3$) which are continuous variables arbitrarily constructed by the observer to interpolate between different events. The assignment of coordinates is based on information derived mainly from the observation of photons, and to a lesser but increasingly important extent from neutrinos transmitted from distant sources. The information conveyed by any such particle concerns its existence, its spin, and its energy, from which the fact of its creation is inferred, together with its direction of incidence and frequency. In context with related observations, the direction of incidence and shift of frequency convey information concerning the nature of the event, the velocity, gravitational potential, and distance of the source. Since the primary information is quantal in nature, the position of the source in particular may be represented by a quantal tape of the form

$$n(z) = \prod_{r=1}^{\sigma} n^{(r)} n^{(\sigma+r)} \ldots n^{(u\sigma+r)} q^{(r)}(z), \qquad q^{(r)}(z) = \tfrac{1}{2}(1 + iz^k \alpha_k^{(r)}) \quad (55)$$

depending on a vector z with components z^k ($k = 1, \ldots, d$, where $d = 2u + 5$) and satisfying

$$\bar{z}z = z^{tr}\,\eta z = 1, \qquad (z^{tr}\eta)_k = z_k = \eta_{kl} z^l \quad (56)$$

The tape is constructed symmetrically from σ sets of anticommuting matrices $\alpha_k^{(r)}$, and must include σ real qubits to obtain an informational metric η_{kl} with signature $(-1, 1 \ldots 1)$ in d dimensions. The z^k can be interpreted as parameters of the continuous group of transformations with elements of the type

$$u(z', z) = \exp(\tfrac{1}{4} z^k z'^l [\alpha_k, \alpha_l]) \quad (57)$$

taking $n(z)$ to $n(z')$. The spinor factors $\chi(z)$ and $\bar{\chi}(z)$ of $n(z)$ defined as in (49) undergo the corresponding transformations

$$\chi(z') = u(z', z)\chi(z), \qquad \bar{\chi}(z') = \bar{\chi}(z) u^{-1}(z', z) \quad (58)$$

and the parameters can in turn be expressed in terms of $\bar{\chi}(z)[\alpha_k, \alpha_l]\chi(z)$.

The separation s of two points z and z' is given in general by

$$s^2 = (\bar{z} - \bar{z}')(z - z') = 2(1 - \bar{z}z') \quad (59)$$

but reduces to ds, where $ds^2 = d\bar{z}dz$ if $z' = z + dz$ is sufficiently near to z. When points of space-time are assigned coordinates x^λ in the manner described, the vector z becomes a function $z(x)$ of the coordinates, and $dz = z_\lambda dx^\lambda$, where $z_\lambda = \partial z / \partial x^\lambda$. It follows that

$$ds^2 = g_{\lambda\mu}\, dx^\lambda\, dx^\mu, \qquad g_{\lambda\mu} = \bar{z}_\lambda z_\mu = z^k_\lambda \eta_{kl} z^l_\mu \quad (60)$$

so that $g_{\lambda\mu}$ is the Riemannian metric tensor, with a contravariant form $g^{\lambda\mu}$ defined by $g^{\lambda\nu} g_{\nu\mu} = \delta^\lambda_\nu$. The Christoffel affinity, defined in the usual way, is $\Gamma^{\lambda}_{\mu\nu} = \bar{z}^{\lambda} z_{\mu,\nu}$, where $\bar{z}^{\lambda} = g^{\lambda\mu}\bar{z}_{\mu}$ and $z_{\mu,\nu}$ is the ordinary derivative of z_{μ} with respect to x^{ν}. It follows that the covariant derivatives of z_{μ} and $\bar{z}_{\mu}$ are given by

$$z_{\mu/\nu} = (1 - \tilde{z}) z_{\mu,\nu} = \tilde{z}_{\nu} z_{\mu}, \qquad \bar{z}_{\mu/\nu} = \bar{z}_{\mu}\tilde{z}_{\nu}, \qquad \tilde{z} = z^{\lambda}\bar{z}_{\lambda} \tag{61}$$

where $z^{\lambda}\bar{z}_{\lambda}$ denotes the outer (matrix) product, and is easily seen to be a projection. The Riemannian curvature tensor may be defined by

$$R^{\rho}_{\lambda\mu\nu} = \bar{z}^{\rho}(z_{\lambda/\mu/\nu} - z_{\lambda/\nu/\mu}) = \bar{z}^{\rho}_{/\mu} z_{\lambda/\nu} - \bar{z}^{\rho}_{/\nu} z_{\lambda/\mu} \tag{62}$$

In its passage between its source and a point of observation, a particle traverses empty space. Einstein's field equation for empty space, with the usual cosmological constant κ, is

$$R_{\lambda\mu} = -R^{\nu}_{\lambda\mu\nu} = \kappa g_{\lambda\mu} \tag{63}$$

where if $\kappa = 3/R^2$, R is of the order of the distance of the cosmic horizon. With the help of (62), (63) can be written entirely in terms of the vector z and its covariant derivatives:

$$\bar{z}^{\nu}_{/\nu} z_{\lambda/\mu} - \bar{z}^{\nu}_{/\mu} z_{\lambda/\nu} = \bar{z}_{\lambda}\kappa z_{\mu} \tag{64}$$

where κ may now be a matrix in general. As the author has shown elsewhere (Green, 1998), from this starting point, here given an informational basis, Einstein's theory of gravitation and its cosmological extensions can be developed in a remarkably simple form. The gravitational field equations can be derived from the action

$$A_g = \int [\bar{z}^{\lambda}\tilde{z}_{\mu}(\tilde{z}^{\mu} z_{\lambda} - \tilde{z}^{\lambda} z_{\mu}) - \bar{z}^{\lambda}\kappa z_{\lambda}) - \mathrm{tr}(\kappa\tilde{z})]\, d^4x \tag{65}$$

and it is possible to formulate generalizations of Dirac's equation and Maxwell's equations in bispinor form which take account of both gravitational and cosmological effects. The result is to show how these, as well as many other areas of theoretical physics, can be related to quantal information theory.

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